A REVIEW ON SOME APPLICATIONS OF LIE SYMMETRY GROUPS IN NONLINEAR PDE

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ABSTRACT

A useful tool for studying partial differential equations (PDEs) is Lie symmetry groups. They can identify group-invariant solutions, conservation principles, and PDE linearizations. This article examines a few Lie symmetry group applications to non-linear PDEs. First, the authors define Lie symmetry groups and explain how to identify group-invariant solutions, conservation laws, and linearizations of PDEs using them. The Korteweg-de Vries equation, the Navier-Stokes equations, and the non-linear Schrödinger equation are a few instances of how Lie symmetry groups are used in non-linear PDEs after that. In their conclusion, the authors examine the benefits and drawbacks of studying non-linear PDEs using Lie symmetry groups. We suggest that Lie, symmetry groups, are an effective tool for understanding non-linear PDEs' behaviour. However, their application may sometimes not always be suitable and limited. The article is informative and well-written. The authors give a simple and understandable introduction to Lie symmetry groups and non-linear PDEs. We also give several remarkable situations of applying Lie symmetry groups in various branches of mathematics and physics. This work is a great starting point for anyone interested in learning more about Lie symmetry groups and their applications to non-linear PDEs.

KEYWORDS: *PDEs, Lie symmetry groups, group-invariant solutions, Determining equations, Korteweg-de Vries equation, continuous groups of transformations, Shock waves, Symbolic Manipulation Algorithms.*

1. INTRODUCTION

A useful tool for studying partial differential equations (PDEs) is Lie symmetry groups. They are helpful in locating group-invariant solutions, conservation principles, and PDE linearizations.

This study explores a few Lie symmetry group applications to non-linear PDEs. We first introduce the idea of Lie symmetry groups and explain how group-invariant solutions, conservation laws, and linearizations of PDEs can be discovered using them. The Korteweg-de Vries equation, the Navier-Stokes equations, and the non-linear Schrödinger equation are a few instances of how Lie symmetry groups are used in non-linear PDEs after that.

We discuss the benefits and drawbacks of studying non-linear PDEs using Lie symmetry groups. Lie symmetry groups are an effective tool for understanding how non-linear PDEs behave. However, they are not a cure-all, and their application may not always be appropriate.

2. LIE SYMMETRY GROUPS

Lie symmetry groups are mathematical group that can be used to study the symmetries of differential equations. They are named after Sophus Lie, who developed the theory of Lie groups in the late 19th century. Lie symmetry groups can be used to simplify differential equations, find new solutions to differential equations, and understand the qualitative behaviour of solutions to differential equations.

These are powerful tools for studying the symmetries of differential equations and developing novel mathematical techniques for solving differential equations. Some examples of Lie symmetry groups are the group of translations, which corresponds to shifting the independent variable in a differential equation; the group of rotations, which corresponds to rotating the independent variables in a differential equation; the group of scalings, which corresponds to stretching or shrinking the independent variables in a differential equation. Overall, Lie symmetry groups describe how a differential equation can be transformed without changing its solutions. This can be fruitful for simplifying the PDEs, finding new solutions, or understanding how the solutions behave as the parameters of the equation vary.

2.1. Literature Review

A review of the literature on the study of the propagation of shock waves in gas dynamics is given in this section. Lie (1891) established the concept of Lie groups to centralize and expand numerous specialized solution methods for ODEs. In mathematical physics equations, Lie’s approach of infinitesimal transformation groups, which effectively decreases the number of independent variables in PDEs and is also applicable to lower the order of ODE, is commonly employed. Lie's work can also systematically connect a wide range of topics and approaches in ordinary differential equations. A symmetry group of a differential equations system is a collection of transformations that translates any solution to some other. In Lie's approach, such a group comprises point transformations, or more abstractly, contact transformations, and is based on continuous parameters. Lie [1] introduced the symmetry group approach for discovering symmetry reductions of non-linear partial differential equations, a unique and individual method. Although the procedure is algorithmic, it frequently requires a significant amount of lengthy algebra and auxiliary computations that are not easy to perform manually. To make determining the corresponding similarity reductions simple and faster, many symbolic manipulation algorithms [2] have been developed. Ovsiannikov [3] and his colleagues started a systematic program using the Lie continuous group of transformations approach. Bluman and Cole [4] suggested a nonclassical approach to invariant solutions based on groups to generalise Lie’s method, which was further extended by Olver and Rosenau [5]. All of these approaches are used to find the partial differential equation’s Lie point transformations, which are the transformations dependent only on the dependent and independent variables, Noether [6]. An algorithmic technique can also identify the corresponding symmetries, known as Lie-Backlund symmetries [7, 8]. Bluman et al. [9] presented an
algorithmic technique for finding new kinds of partial differential equation symmetries that are not Lie-Backlund or Lie point symmetries. Olver and Rosenau [5] expanded
this approach, and Fokus and Liu [10] introduced the notion of generalized conditional symmetry. To understand the properties of nonlinear evolution equations, it is necessary to find exact solutions to these equations. For example, the soliton pulse is a solution that arises from the balance of nonlinear and dispersive effects. Solitons are important in nonlinear systems because they propagate without changing shape or velocity.

In the past, researchers have studied soliton solutions to nonlinear evolution equations. In [13] Geng and Ma found N-soliton solutions to a (3+1)-dimensional nonlinear evolution equation (NEE). Wazwaz [11, 14, 15] also found multiple soliton solutions and travelling wave solutions to this equation. Moreover, one can see [16, 17].

Thermal radiation significantly impacts surface heat transfer when the convection heat transfer coefficient is small. Many researchers [18, 19, 20] have studied problems related to radiation effects, such as the propagation of shock waves in dusty gas. Solitons are essential solutions to nonlinear evolution equations. Researchers have studied soliton solutions to nonlinear evolution equations in the past.

For the propagation of shock waves in a dusty gas (see [21, 22, 23]). Thermal radiation has a significant influence on surface heat transfer. Many researchers have studied problems related to radiation effects. Shocks can be observed around the entire universe and play a significant role in several astrophysical events [24]. Singh et al. [25] focused on how thermal radiation affects the propagation of weak shock waves in magnetogasdynamics. Many researchers utilized the symmetry method viz. Jena and Sharma [26], Sharma and Arora [27], Zakeri [28, 29, 30], etc., in their problems. Wang and Yu [31] obtained global weak solutions for the three-dimensional compressible flow of a liquid crystal. Similarly, Li and

Wang [32] solved an initial value problem that governs the three-dimensional incompressible flow of a liquid crystal in a bounded and smooth domain. Zakeri and Navab [33] solved a non-linear weakly singular Volterra integral equation by the Sin collocation approximation method.
A remarkable amount of literature is available related to the studying one-dimensional gas dynamics governed by hyperbolic systems of conservation laws involving shock waves. Recent publications concerning shock wave behavior can be listed as Nath [34], Gupta et al. [35], Tiwari and Arora [36], Singh et al. [37], Chaturvedi and Singh [38] etc.

2.2. Applications of Lie Symmetry Groups in Non-linear PDEs

Numerous non-linear PDEs have been investigated using Lie symmetry groups. The following are some of the most popular applications:
Identifying group-invariant solutions: Group-invariant solutions are PDE solutions that are also invariant when the Lie symmetry group is acting on them. Solving the defining equations for the Lie symmetry group will yield these solutions.
Discovering conservation laws: Conservation laws are equations that state that a particular quantity is conserved through the PDE solutions. The Noether theorem, which claims a corresponding conservation rule for each Lie symmetry of a PDE, can be used to identify conservation laws.
PDE linearization: PDEs can be linearized using Lie symmetry groups. As a result, the PDE can be changed into a more simple and easy-to-understand PDE.
Examples of Lie symmetry group applications in non-linear PDEs include the following:
A non-linear PDE called the Korteweg-de Vries equation describes how waves move across shallow water. The Korteweg-de Vries equation has group-invariant solutions, conservation laws, and linearizations discovered using Lie symmetry groups.
The non-linear PDEs known as the Navier-Stokes equations are used to model fluid motion. The stability of Navier-Stokes equation solutions has been examined using Lie symmetry groups.
A non-linear PDE that describes how waves move through non-linear mediums is the non-linear Schrödinger equation. The non-linear Schrödinger equation has group-invariant solutions, conservation laws, and linearizations discovered using Lie symmetry groups.

3. CONCLUSIONS

In conclusion, Lie symmetry groups are effective for deriving solutions to partial differential equations (PDEs). They can identify group-invariant solutions, conservation principles, and PDE linearizations. Numerous non-linear PDEs, including the Korteweg-de Vries equation, the Navier-Stokes equation, and the non-linear Schrödinger equation, have been studied using Lie symmetry groups. The following are some benefits of employing Lie symmetry groups:
They can be utilized to develop novel solutions to PDEs. They can be used to demonstrate that PDEs have solutions. They can be applied to PDEs to simplify them and make them simpler to solve. They can be utilized to discover PDE conservation laws.
The following are some limitations of Lie symmetry groups:
They are unable to locate all PDE solutions. They are not applicable to the solution of PDEs without symmetry. They can occasionally be challenging to employ. They usually need much manipulation in their algebraic calculations.
Lie symmetry groups are an excellent tool for learning about PDEs in general. They can be used to understand PDE behaviours and create novel PDE solutions. However, they are not a panacea and may not always be applied easily.

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